The CPT group of the spin-3/2 field

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Abstract

We find out that both the matrix and the operator CPT groups for the spin-3/2 field (with or without mass) are respectively isomorphic to $D_4 \rtimes \mathbb{Z}_2$ and $Q \times \mathbb{Z}_2$. These groups are exactly the same groups as for the Dirac field, though there is no a priori reason why they should coincide.

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1 Introduction

The CPT group of the Dirac field in Minkowski space-time was obtained by Socolovsky in 2004 [1]. It were found two sets of consistent solutions for the matrices of charge conjugation (C), parity (P), and time reversal (T), which give the transformation of fields $\hat{\psi}_C(x) = C\hat{\bar{\psi}}^T(x)$, $\hat{\psi}_\Pi(x_\Pi) = P\hat{\psi}(x)$ and $\hat{\psi}_\tau(x_\tau) = T\hat{\psi}(x)^*$, where $x_\Pi = (t, -\mathbf{x})$ and $x_\tau = (-t, \mathbf{x})$. These sets are given by:

a) $C_D = \pm\gamma^2\gamma^0$, $P_D = \pm i\gamma^0$, $T_D = \pm i\gamma^3\gamma^1$,
b) $C_D = \pm i\gamma^2\gamma^0$, $P_D = \pm i\gamma^0$, $T_D = \pm\gamma^3\gamma^1$.

Each of these sets generates a non abelian group of sixteen elements, respectively, $G_\theta^{(1)} \cong D_4 \times \mathbb{Z}_2$ and $G_\theta^{(2)} \cong 16E$, where D_4 is the group of symmetries of the square and $16E$ is a non trivial extension of D_4 by $\mathbb{Z}_2$, isomorphic to a semidirect product of these groups.

On the other hand, the quantum operators $\hat{C}$, $\hat{P}$ and $\hat{T}$, acting on the Hilbert space, generate a unique group $G_{\hat{\theta}} \cong Q \times \mathbb{Z}_2$, where Q is the quaternion group.

With this in mind, we decided to find the CPT group of the spin-3/2 field (Rarita-Schwinger field), for both massive and massless cases. This field could be useful for the description of compound objects (neglecting its structure in a first approximation), like the baryon decuplet components for spin-$3/2^+$ [2], or for elementary fields such as the gravitino.

In order to describe 3/2-spin particles, the set of equations

$$(i\gamma^\alpha\partial_\alpha - m)\hat{\psi}^\mu(x) = 0, \tag{1}$$

$$\gamma^\mu\hat{\psi}_\mu(x) = 0, \tag{2}$$

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, is required. These equations are known as the Rarita-Schwinger equation [3]. The first of these equation is a Dirac equation for each vector component of the vector spinor $\hat{\psi}^\mu$ and the second one is known as the subsidiary condition. Precisely due to the more complexity of these equations with respect to the Dirac field equation, there is not a priori any apparent reason why the CPT group for the Rarita-Schwinger field should coincide with those for the Dirac field.

2 Parity

If we want to study the P invariance of the Rarita-Schwinger equation, we need to repeat the analysis done for the Dirac equation in [1] and also consider the P invariance of the subsidiary equation.

Multiplying this equation from the left by P, changing $\mathbf{x} \rightarrow -\mathbf{x}$ and inserting the unity, one obtains:

$$\begin{aligned} P\gamma^0 P^{-1} P\hat{\psi}_0(t,-\mathbf{x}) + P\gamma^1 P^{-1} P\hat{\psi}_1(t,-\mathbf{x}) + P\gamma^2 P^{-1} P\hat{\psi}_2(t,-\mathbf{x}) \\ + P\gamma^3 P^{-1} P\hat{\psi}_3(t,-\mathbf{x}) = 0, \end{aligned} \tag{3}$$

but we need to take into account that the vector spinor changes by parity in the following way:

$$\hat{\psi}_\mu(t,\mathbf{x}) = \begin{pmatrix} \hat{\psi}_0(t,\mathbf{x}) \\ \hat{\psi}_1(t,\mathbf{x}) \\ \hat{\psi}_2(t,\mathbf{x}) \\ \psi_3(t,\mathbf{x}) \end{pmatrix} \longrightarrow \hat{\psi}_{\mu\pi}(t,\mathbf{x}) = \begin{pmatrix} P\hat{\psi}_0(t,-\mathbf{x}) \\ -P\hat{\psi}_1(t,-\mathbf{x}) \\ -P\hat{\psi}_2(t,-\mathbf{x}) \\ -P\psi_3(t,-\mathbf{x}) \end{pmatrix}; \tag{4}$$

where $\hat{\psi}_{\mu\pi}(t,\mathbf{x})$ can be written as $\hat{\psi}_{\mu\pi}(t,\mathbf{x}) = \mathcal{P}P\hat{\psi}_\mu(t,-\mathbf{x})$, with $\mathcal{P} \in O(1,3)$ given by:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

and $P \in D^{16}$, where D^{16} is the Dirac algebra.

Substituting the components of $\hat{\psi}_{\mu\pi}(t,\mathbf{x})$ in (3) one obtains:

$$P\gamma^0 P^{-1}\hat{\psi}_{0\pi}(t,\mathbf{x}) - P\gamma^k P^{-1}\hat{\psi}_{k\pi}(t,\mathbf{x}) = 0. \tag{6}$$

This implies the constraints on P:

$$P\gamma^0 P^{-1} = \gamma^0, \quad P\gamma^k P^{-1} = -\gamma^k, \tag{7}$$

or

$$P\gamma^0 P^{-1} = -\gamma^0, \quad P\gamma^k P^{-1} = \gamma^k. \tag{8}$$

The relations (7) are the same as those obtained from the Dirac equation, whose already known solution is $P_D = \pm i\gamma^0$, while from the relations (8) is obtained $P = P' = z\gamma^3\gamma^2\gamma^1$, with $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Following the same analysis as in [1], there are two possibilities for each P:

$$\text{a) } P^2 = +1 \;\Rightarrow\; z = \pm 1 \;\Rightarrow P' = \pm\gamma^3\gamma^2\gamma^1;$$
$$\text{b) } P^2 = -1 \;\Rightarrow\; z = \pm i \;\Rightarrow P' = \pm i\gamma^3\gamma^2\gamma^1.$$

In the first case: $P'^{\dagger} = P' = P'^{-1} = -P'^{T} = -P'^{*}$ and in the second one: $P'^{\dagger} = -P' = P'^{-1} = P'^{T} = -P'^{*}$.

3 Charge conjugation

To study the C invariance of the subsidiary equation, we must take the complex conjugate of this equation, multiply from the left by $C\gamma_0$ and insert the unit matrix. This is:

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1}\hat{\psi}_{\mu C}(x) = 0, \tag{9}$$

where $\hat{\psi}_{\mu C}(x) = C\gamma^0\hat{\psi}^{*}_{\mu}(x)$.

In this case, the constraints on C are:

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = \gamma^{\mu}, \tag{10}$$

or

$$(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1} = -\gamma^{\mu}. \tag{11}$$

Taking into account that $\gamma^0\gamma^{\mu *}\gamma^0 = \gamma^{\mu T}$, one can find the solutions for the C matrices. The equation with the negative sign is the same as in the Dirac case and leads to $C = C_D = \eta\gamma^2\gamma^0$; on the other hand, from equation (10) we arrive to:

$$C\gamma^{\mu T}C^{-1} = \gamma^{\mu}, \tag{12}$$

which leads to the solution $C = C' = \eta\gamma^3\gamma^1$.

For C', a second application of the charge conjugation transformation is given by:

$$\begin{aligned}(\hat{\psi}_C)_C &= \hat{\psi}_{C^2} = C'\hat{\bar{\psi}}^{T}_{C} = C'(\hat{\psi}^{\dagger}_{C}\gamma^0)^{T} = C'\gamma^0\hat{\psi}^{*}_{C} = C'\gamma^0 C'^{*}\gamma^0\hat{\psi} = -C'C'^{*}\hat{\psi} \\ &= -|\eta|^2\gamma^3\gamma^1\gamma^{3*}\gamma^{1*}\hat{\psi} = |\eta|^2(\gamma^3)^2(\gamma^1)^2\hat{\psi} = |\eta|^2\hat{\psi}.\end{aligned} \tag{13}$$

Since the effect on $\hat{\psi}$ can be, at most, a multiplication by a phase, then $\eta \in U(1)$ and C' is unitary. This is:

$$C^{'}C^{'\dagger} = \eta\gamma^3\gamma^1\bar{\eta}(\gamma^3\gamma^1)^\dagger = |\eta|^2\gamma^3\gamma^1\gamma^1\gamma^3 = |\eta|^2\gamma^3(\gamma^1)^2\gamma^3 = |\eta|^2 1 = 1. \quad (14)$$

Hence, for $C = C^{'}$ we also find that $\hat{\psi}_{C^2} = \hat{\psi}$.

As in [1], due to a symmetry consideration between $\hat{\psi}_C = C^{'}\hat{\bar{\psi}}^T$ and $\hat{\bar{\psi}}_C = -\bar{\eta}^2\hat{\psi}^T C^{'}$, it follows that $-\bar{\eta}^2 = \pm 1$ and taking into account that $|\eta|^4 = 1$, one obtains $\eta^2 = \pm 1$, which implies that $\eta = \pm 1, \pm i$.

Then, for the spin-3/2 field there are two possibilities for each C:

a) $C^{'} = \pm\gamma^3\gamma^1, \qquad C_D = \pm\gamma^2\gamma^0$;
b) $C^{'} = \pm i\gamma^3\gamma^1, \qquad C_D = \pm i\gamma^2\gamma^0$.

4 Time reversal

We start again from the subsidiary equation, change $t \to -t$ and take the complex conjugate:

$$T\gamma^{0*}T^{-1}T\hat{\psi}_0(-t,\mathbf{x})^* + T\gamma^{1*}T^{-1}T\hat{\psi}_1(-t,\mathbf{x})^* + T\gamma^{2*}T^{-1}T\hat{\psi}_2(-t,\mathbf{x})^* + T\gamma^{3*}T^{-1}T\hat{\psi}_3(-t,\mathbf{x})^* = 0, \quad (15)$$

but we need to take into account that the vector spinor changes by time inversion in the following way:

$$\hat{\psi}_\mu(t,\mathbf{x}) = \begin{pmatrix} \hat{\psi}_0(t,\mathbf{x}) \\ \hat{\psi}_1(t,\mathbf{x}) \\ \hat{\psi}_2(t,\mathbf{x}) \\ \hat{\psi}_3(t,\mathbf{x}) \end{pmatrix} \longrightarrow \hat{\psi}_{\mu\tau}(t,\mathbf{x}) = \begin{pmatrix} -T\hat{\psi}_0(-t,\mathbf{x})^* \\ T\hat{\psi}_1(-t,\mathbf{x})^* \\ T\hat{\psi}_2(-t,\mathbf{x})^* \\ T\hat{\psi}_3(-t,\mathbf{x})^* \end{pmatrix}; \quad (16)$$

where $\hat{\psi}_{\mu\tau}(t,\mathbf{x})$ can be written as $\hat{\psi}_{\mu\tau}(t,\mathbf{x}) = \mathcal{T}T\hat{\psi}_\mu(-t,\mathbf{x})^*$, with $\mathcal{T} \in O(1,3)$ given by:

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (17)$$

and $T \in D^{16}$.

Substituting the components of $\hat{\psi}_{\mu\tau}(t,\mathbf{x})$ in (15) one obtains:

$$-T\gamma^{0*}T^{-1}\hat{\psi}_{0\tau}(t,\mathbf{x})+T\gamma^{k*}T^{-1}\hat{\psi}_{k\tau}(t,\mathbf{x})=0, \tag{18}$$

from which we can deduce the constraints on T:

$$T\gamma^{0}T^{-1}=\gamma^{0}, \quad T\gamma^{k*}T^{-1}=-\gamma^{k}, \tag{19}$$

or

$$T\gamma^{0}T^{-1}=-\gamma^{0}, \quad T\gamma^{k*}T^{-1}=\gamma^{k}. \tag{20}$$

The relations (19) are the same as those obtained from the Dirac equation, whose already known solution is $T_D=e^{i\lambda}\gamma^3\gamma^1$, while from the relations (20) is obtained $T=T^{'}=w\gamma^2\gamma^0$, with $w\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$.

For $T=T^{'}$, applying τ twice, we arrive to:

$$\hat{\psi}(t,\mathbf{x})\to\hat{\psi}_{\tau}(t,\mathbf{x})=T^{'}\hat{\psi}(-t,\mathbf{x})^*\to T^{'}(T^{'}\hat{\psi}(t,\mathbf{x})^*)^*=T^{'}T^{'*}\hat{\psi}(t,\mathbf{x}). \tag{21}$$

and taking into account that

$$T^{'}T^{'*}=|w|^2\gamma^2\gamma^0(\gamma^2\gamma^0)^*=-|w|^2\gamma^2\gamma^0\gamma^2\gamma^0=|w|^2\gamma^2\gamma^2\gamma^0\gamma^0=-|w|^2 1, \tag{22}$$

it follows that $\hat{\psi}_{\tau^2}=-\hat{\psi}$, by a similar argument to that used for C.

Thus, $T^{'}T^{'*}=-1$, which implies $T^{'*}=-T^{'-1}$ and $w\in U(1)$. Then, $T^{'}=e^{i\lambda}\gamma^2\gamma^0$ y $T^{'\dagger}=e^{-i\lambda}\gamma^2\gamma^0$.

5 Matrix and operator CPT groups

In summary, one has both the two sets of matrices:

a) $C_D=\pm\gamma^2\gamma^0, \qquad P_D=\pm i\gamma^0, \qquad T_D=\pm i\gamma^3\gamma^1,$
b) $C_D=\pm i\gamma^2\gamma^0, \qquad P_D=\pm i\gamma^0, \qquad T_D=\pm\gamma^3\gamma^1;$

and the set: $C^{'},P^{'},T^{'}$, satisfying the subsidiary equation. But, only the matrices C_D,P_D,T_D, also satisfy the Dirac type equation. That is why they are the matrices which conform the matrix CPT group of the spin-3/2 field with mass.

If we take the zero mass limit of the Dirac type equation,

$$i\gamma^\alpha\partial_\alpha\hat{\psi}(x)=0, \tag{23}$$

and analyze its behavior under parity, charge conjugation and time reversal, as we did for the subsidiary condition, we have, respectively:

$$i(P\gamma^0 P^{-1}\partial_0 - P\gamma^i P^{-1}\partial_i)\hat{\psi}_\Pi(x) = 0, \tag{24}$$

$$(i\partial_\mu + qA_\mu)(C\gamma^0)\gamma^{\mu *}(C\gamma^0)^{-1}\hat{\psi}_C(x) = 0. \tag{25}$$

$$i(\gamma^{0*}\frac{\partial}{\partial t} - \gamma^{k*}\frac{\partial}{\partial x^k})\hat{\psi}_\tau(x) = 0. \tag{26}$$

From the above equations we can then obtain the corresponding restrictions on the C, P and T matrices, respectively:

$$\begin{aligned} P\gamma^0 P^{-1} = \pm\gamma^0, \quad P\gamma^k P^{-1} = \mp\gamma^k, \\ C\gamma^{\mu T} C^{-1} = \pm\gamma^\mu, \\ T\gamma^0 T^{-1} = \pm\gamma^0, \quad T\gamma^{k*} T^{-1} = \mp\gamma^k. \end{aligned} \tag{27}$$

These relations generate the same sets of matrices C_D, P_D, T_D and $C^{'}, P^{'}, T^{'}$ which gave the subsidiary condition. But if we take into account that $\hat{\bar{\psi}}\hat{\psi}$ is the charge density operator, for $C = C^{'}$, it follows that:

$$\begin{aligned} (\hat{\bar{\psi}}\hat{\psi})_C = \hat{\bar{\psi}}_C\hat{\psi}_C = -\bar{\eta}^2\hat{\psi}^T C^{'} C^{'}\hat{\bar{\psi}}^T = -\bar{\eta}^2 C^{'2}(\hat{\bar{\psi}}\hat{\psi})^T \\ = -\bar{\eta}^2 C^{'2}\hat{\bar{\psi}}\hat{\psi} = (\bar{\eta}\eta)^2\hat{\bar{\psi}}\hat{\psi} = |\eta|^4\hat{\bar{\psi}}\hat{\psi} = -\hat{\bar{\psi}}\hat{\psi}; \end{aligned} \tag{28}$$

from which $|\eta|^4 = -1$, which is a contradiction. Hence, the matrix $C = C^{'}$ must be discarded.

It was demostrated in [1] that C and P must fulfill the relation:

$$C(P^{-1})^T C^{-1} = P, \tag{29}$$

while C and T are related by:

$$CT^* = TC^*. \tag{30}$$

Due to the fact that each component of the vector spinor satisfies a Dirac type equation, the above relations also hold in our case. That is why the set of matrices: $C^{'}, P^{'}, T^{'}$ are also discarded for the massless case.

In order to find the operator CPT group for the spin-3/2 field (with or without mass), we follow the same procedure developed in [1], for the case of the corresponding CPT group of the Dirac field.

Taking $\hat{A}$ and $\hat{B}$ as any of the operators $\hat{C}_D$, $\hat{P}_D$ and $\hat{T}_D$; and $\hat{\psi}$, as each component of the vector spinor $\hat{\psi}^\mu(x)$, the relations:

$$\hat{A} \cdot \hat{\psi} = \hat{A}^\dagger \hat{\psi} \hat{A} \tag{31}$$

and

$$(\hat{A} * \hat{B}) \cdot \hat{\psi} = (\hat{A}\hat{B})^\dagger \hat{\psi} (\hat{A}\hat{B}), \tag{32}$$

can be defined.

Using the above expressions and with support in the matrix CPT group, through the formulas that link matrices with operators:

$$\begin{aligned} P\hat{\psi}(t, -\mathbf{x}) &= \hat{P}^\dagger \hat{\psi}(t, \mathbf{x}) \hat{P}, \\ C\hat{\bar{\psi}}^T(x) &= \hat{C}^\dagger \hat{\psi}(x) \hat{C}, \\ T\hat{\psi}(-t, \mathbf{x})^* &= \hat{T}^\dagger \hat{\psi}(t, \mathbf{x})^\dagger \hat{T}; \end{aligned} \tag{33}$$

the relations:

$$\begin{gathered} \hat{P}_D * \hat{P}_D = -1, \quad \hat{C}_D * \hat{C}_D = 1, \quad \hat{T}_D * \hat{T}_D = -1, \\ \hat{T}_D * \hat{P}_D = -\hat{P}_D * \hat{T}_D, \; \hat{C}_D * \hat{P}_D = \hat{P}_D * \hat{C}_D, \; \hat{C}_D * \hat{T}_D = \hat{T}_D * \hat{C}_D, \end{gathered} \tag{34}$$

were obtained in [1]; from which it is possible to build, also using the property of associativity, the multiplication table for the operator CPT group.

It was also demostrated in [1] that only the second of the two solutions for the matrix group ($G_\theta^{(2)} \cong 16E \cong D_4 \rtimes \mathbb{Z}_2$), is compatible with the operator group ($G_{\hat{\theta}} \cong Q \times \mathbb{Z}_2$).

In summary, we showed that both the matrix and the operator CPT groups for the spin-3/2 field (with or without mass) coincide with the corresponding groups for the Dirac field.

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